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LETTER TO THE EDITOR

An interpretation of 2D quasiperiodic patterns in terms of automorphisms of free groups

Juan Garcia-Escudero† and Peter Kramer‡

† Departamento de Física, Universidad de Oviedo, 33007 Oviedo, Spain
‡ Institut für Theoretische Physik, Universität Tübingen, 72076 Tübingen, Germany

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Abstract. The Penrose and triangle patterns are interpreted as strings in the context of free grammars. The set of production rules of such grammars is induced by certain automorphisms of free groups.

1. Introduction

Sequences like the Fibonacci or Thue–Morse system may be viewed in the field of combinatorics on words. Given a (finite) set called an alphabet $\Sigma=\langle x_1, \ldots, x_n\rangle$, any finite sequence formed from elements of Σ is called a word w. The set of all words is denoted by Σ^*. The empty word e is defined by $ew=we=w$. If the multiplication of words is defined by concatenation, the set Σ^* including the empty word becomes a free monoid and the set $\Sigma^+=\Sigma^*-1$ a free semigroup. A morphism ϕ from a monoid Σ^* to another monoid and the set Σ'^* is defined by the properties $\phi(w_1w_2)=\phi(w_1)\phi(w_2)$ for any two words $w_1, w_2\in\Sigma^*$, and $\phi(e)=e'$. To specify a morphism it clearly suffices to give the images of the alphabet Σ.

The monoid with alphabet Σ is extended into the free group F_n by formally introducing inverses with the properties $x_i^{-1}: x_i^{-1}x_i=x_ix_i^{-1}=e$, $1\leqslant i\leqslant n$ [1]. Now the alphabet is called the generating set. An automorphism σ is an invertible morphism $\sigma: F_n\rightarrow F_n$. *The set of all invertible morphisms of the free group F_n is a group denoted by Φ_n.* This automorphism group was characterized in general by Nielsen, who also gives a finite set of generators for Φ_n [2–4]. The Fibonacci strings may be interpreted as a particular set of words from Σ^* based on an alphabet $\Sigma=\langle x_1, x_2\rangle$. The standard local inflation rule, written as

$$\begin{aligned}\sigma(x_1)&=x_2\\ \sigma(x_2)&=x_1x_2\end{aligned}\tag{1}$$

by iteration generates the Fibonacci strings [5]. The map σ has an inverse and so becomes an automorphism of F_2. Hence the (infinite) Fibonacci string is the orbit starting at x_1 under the positive powers of the automorphism σ. In the terminology used for quasicrystals, general words correspond to random tilings and orbits to ordered tilings.

The main purpose of this paper is to extend this interpretation of 1D quasiperiodic systems to the 2D case. In order to do this we need to implement the automorphism with a parenthesis structure which can be introduced by using concepts of formal language theory. A formal grammar is a quadruple

$$G=\langle \Sigma_T, \Sigma_N, S, P\rangle \tag{2}$$

where Σ_T is the alphabet of terminal symbols and Σ_N the alphabet of non-terminal symbols, $S\in\Sigma_N$ is the start symbol or sentence symbol and P is a finite set of production (or substitution) rules which are induced by the specific automorphism. While productions may be composed arbitrarily of terminals and non-terminals, the specified language contains words of terminals only. The specific grammars we will use are context-free grammars or grammars of type 2 [6].

2. The triangle pattern

The triangle pattern *TP* was constructed in [7] by projection from the Delaunay cells of the root lattice A_4. We shall use a specific inflation process for this pattern given by

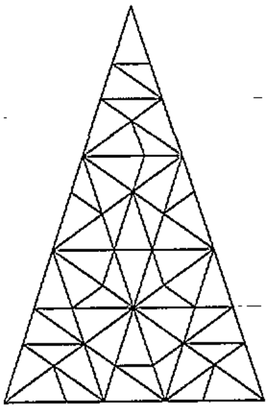

Figure 1. A piece of the triangle pattern.

Schlottmann [8] and shown in figure 1. Now we shall interpret the triangle pattern as a word: a sequence formed from elements of the alphabet $\Sigma=\langle a_i, b_i, a_i', b_i'\rangle$ where $i\in Z_{10}$. The elements a_i, a_i' on one hand and b_i, b_i' on the other hand represent tiles with the same shape but they are distinguished by a colour: white for a_i and b_i, black for a_i' and b_i'. The tile a_i is an acute isosceles triangle with (*length of side*)/(*length of base*) = $\tau=(1+\sqrt{5})/2$, and the tile b_i is an obtuse triangle with (*length of side*)/(*length of base*) = $1/\tau$. The sides of b_i have the same length as the basis of a_i.

In figure 2 we have drawn the tiles a_1 and b_1. The tile a_i (b_i) is obtained from a_1 (b_1) by a rotation of angle $2\pi(i-1)/10$ through the vertex indicated.

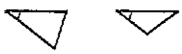

Figure 2. The two tiles a_1 (acute) and b_1 (obtuse) of the triangle pattern.

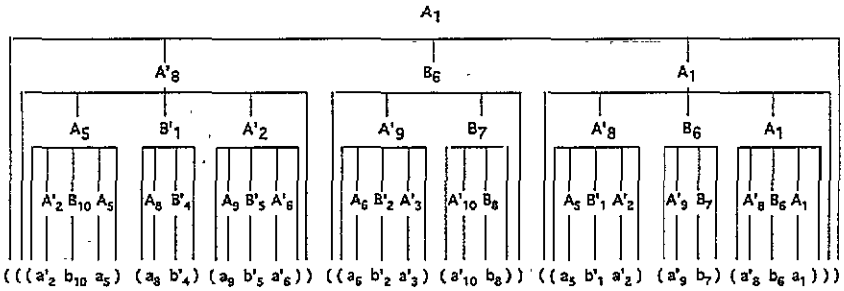

Figure 3. Tree structure for the triangle pattern.

Consider now the free group $F_{40}=\langle a_i, b_i, a_i', b_i', i\in Z_{10}\rangle$ and the corresponding automorphism group Φ_{40}. Let $\omega\in\Phi_{40}$ be defined in the following fashion:

$$\begin{aligned}
\omega(a_i) &= a'_{i-3}b_{i+5}a_i \\
\omega(b_i) &= a'_{i+3}b_{i+1} \\
\omega(a'_i) &= a_{i-3}b'_{i+3}a'_{i+4} \\
\omega(b'_i) &= a_{i-3}b'_{i+3}.
\end{aligned} \tag{3}$$

This automorphism has the inverse

$$\begin{aligned}
\omega^{-1}(a_i) &= b_{i+4}^{-1}a_i \\
\omega^{-1}(b_i) &= a'^{-1}_{i-2}b'_{i-2}b_{i-1} \\
\omega^{-1}(a'_i) &= b'^{-1}_{i-4}a'_{i-4} \\
\omega^{-1}(b'_i) &= a^{-1}_{i+4}b_{i-2}b'_{i-3}.
\end{aligned} \tag{4}$$

We convert the first step of this automorphism into a gluing of triangles by the following prescription: if two letters follow one another, the corresponding oriented triangles are glued face-to-face. This prescription is unique, as can be seen from figure 4. In the second application of the automorphism we use this prescription for the new triangles, which are scaled by a factor τ, and glue them face-to-face, disregarding their internal composition into the initial tiles (figure 4). In the words of F_{40} produced in this second step, not all consecutive letters correspond to glued tiles. To characterize the geometric hierarchy of this gluing process in the word structure, we introduce a formal grammar G^{TP} which we define as follows.

We take the alphabets $\Sigma_T=\langle a_i, b_i, a_i', b_i', (\,,\,)\rangle$, $\Sigma_N=\langle A_i, B_i, A_i', B_i'\rangle$ with $i\in Z_{10}$. Every element belonging to Σ_N can be used as start symbol S. We take $S=A_1$. The set of production rules are

$$\begin{aligned}
P^{TP}=\{&A_i\to(A'_{i-3}B_{i+5}A_i)|a_i,\ B_i\to(A'_{i+3}B_{i+1})|b_i, \\
&A'_i\to(A_{i-3}B'_{i+3}A'_{i+4})|a'_i,\ B'_i\to(A_{i-3}B'_{i+3})|b'_i\}.
\end{aligned} \tag{5}$$

The vertical bars separate two alternative possible images under the map given by the arrows. In figure 3 we give the tree of G^{TP} which represents the implementation by parentheses of the orbit under ω^3 starting at a_1. The corresponding parts of the triangle pattern are shown in figure 4. The fourth generation is represented in figure 1. We

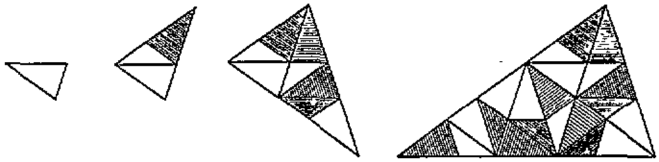

Figure 4. The triangle pattern: orbits under $\omega^n (n=0, 1, 2, 3)$ starting at a_1.

conclude that the grammar structure produces the correct geometric generation of a triangle pattern.

The automorphism ω and hence the set of production rules are not unique. For instance we can take

$$\begin{aligned}
\omega'(a_i) &= a_i b_{i+5} a'_{i-3} \\
\omega'(b_i) &= b_{i+1} a'_{i+3} \\
\omega'(a'_i) &= a'_{i+4} b'_{i+3} a_{i-3} \\
\omega'(b'_i) &= b'_{i+3} a_{i-3}.
\end{aligned} \tag{6}$$

Observe that not all the words belonging to the language generated by the grammar characterize pieces of the pattern; only those appear with the same derivation length for the elements of $\Sigma \cap \Sigma_T$.

3. The Penrose pattern

We will now describe the so-called Robinson decomposition of the Penrose pattern [9] denoted by *PP*. We choose a particular pattern with bilateral symmetry, see figure 5, and use the same basic tiles as in the triangle pattern, see also figure 2: ten white acute triangles a_i, ten white obtuse b_i, ten black acute a'_i and ten black obtuse b'_i. A specific inflation process is now characterized by the following automorphism:

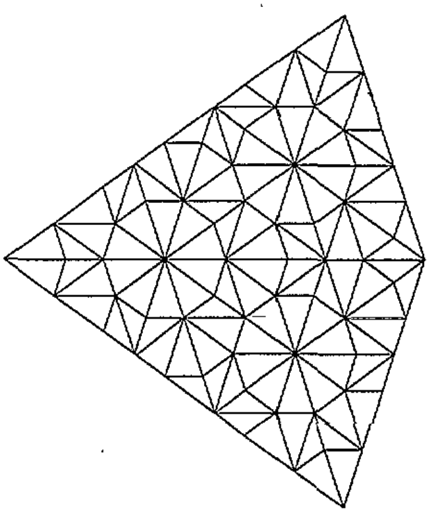

Figure 5. A piece of the Penrose pattern with bilateral symmetry.

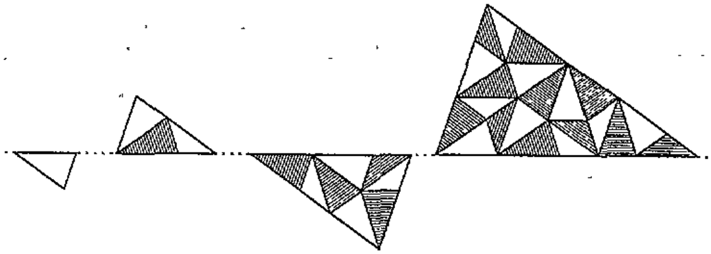

Figure 6. The Penrose pattern: orbits under $\theta^n (n=0,1,2,3)$ starting at a_1.

$$
\begin{aligned}
\theta(a_i) &= a_{i+2}a'_{i+1}b_{i-1} \\
\theta(b_i) &= a'_{i+1}b_{i-1} \\
\theta(a'_i) &= a'_{i-2}a_{i-1}b'_{i+5} \\
\theta(b'_i) &= a_{i-5}b'_{i+1}
\end{aligned}
\tag{7}
$$

with inverse

$$
\begin{aligned}
\theta^{-1}(a_i) &= a_{i-2}b_{i-2}^{-1} \\
\theta^{-1}(b_i) &= b'_{i-2}a'^{-1}_{i+4}b_{i+1} \\
\theta^{-1}(a'_i) &= a'_{i+2}b'^{-1}_{i+6} \\
\theta^{-1}(b'_1) &= b_{i+2}a_{i+2}^{-1}b'_{i-1}.
\end{aligned}
\tag{8}
$$

In this case the pattern is divided into two subpatterns, one of them is described by the odd powers of θ, and the other one by the even powers of θ.

The alphabets are the same as in the triangle pattern, we can take also $S=A_1$, and the set of production rules induced by θ as

$$
\begin{aligned}
P^{PP} = \{ & A_i \to (A_{i+2}A'_{i+1}B_{i-1})|a_i,\ B_i \to (A'_{i+1}B_{i-1})|b_i \\
& A'_i \to (A'_{i-2}A_{i-1}B'_{i+5})|a'_i,\ B'_i \to (A_{i-5}B'_{i+1})|b'_i \}.
\end{aligned}
\tag{9}
$$

In figure 6 we show the orbit of θ^n up to $n=3$. The broken line represents the symmetry axis of the pattern; the odd powers of θ generate the subpattern on top, the even ones the subpattern below the symmetry axis. Observe that in this case two neighbouring tiles with a common edge have always different colours. Also the pairing of two obtuse triangles appears in the subpattern generated by θ^2, while it is forbidden in the triangle pattern.

4. Symmetries of the generating automorphisms

In this section we wish to combine the growth algorithm with five-fold symmetry. Consider the underlying Weyl group of the Lie algebra of type A_4 described by the Coxeter–Dynkin diagram:

The presentation for the Weyl group can be obtained from reading the Coxeter–Dynkin diagram. There is an element r_i ($i=1,2,3,4$) for each node of the diagram which is a reflection in a wall of a spherical simplex. For ADE-type Weyl groups the reading of the Coxeter–Dynkin diagram is particularly simple. Two nodes being joined mean that the corresponding walls are at an angle of $\pi/3$, and if they are not joined, the angle between them is $\pi/2$. The presentation for the group A_4 is

$$\begin{aligned} &r_i^2 = e \qquad i=1,2,3,4 \\ &(r_1r_3)^2=(r_1r_4)^2=(r_2r_4)^2=e \\ &(r_ir_{i+1})^3=e \qquad i=1,2,3. \end{aligned} \tag{10}$$

Now consider the Coxeter group

$$H=\langle R_1, R_2 \,|\, (R_1R_2)^5=R_1^2=R_2^2=e\rangle \tag{11}$$

where $R_1=r_1r_3$ and $R_2=r_2r_4$. We define the following action of H on F_{40}:

$$\begin{array}{llll} R_1(a_i)=a'_{9-i} & R_1(b_i)=a'_{3-i} & R_1(a'_i)=a_{9-i} & R_1(b'_i)=b_{3-i} \\ R_2(a_i)=a'_{7-i} & R_2(b_i)=b'_{1-i} & R_2(a'_i)=a_{7-i} & R_2(b'_i)=b_{1-i}. \end{array} \tag{12}$$

This Coxeter group describes five-fold rotations combined with reflections [5]. See [10] for non-commutative models and Weyl groups as groups of automorphisms.

We claim that ω^n and θ^n are H-invariant, where by H-invariance we mean that R_1 and R_2 commute with ω^n and θ^n modulo a rotation: a shift by the same amount of the indices of all the elements of the string (geometrically it corresponds to a rotation of the whole pattern). First observe that for any generator $x_i \in F_{40}$, $R_1R_2(x_i)=x_{i+2}$, hence R_1R_2 commutes with both ω and θ. In the following, we consider the two automorphisms separately.

For the triangle pattern, comparing the actions of $R_k\omega$ and ωR_k on the generators of F_{40}, it is easy to see that $(R_1R_2)^2R_k\omega=\omega R_k$: $k=1,2$. By induction on n we can prove

$$(R_1R_2)^{2n}R_k\omega^n=\omega^nR_k. \tag{13}$$

To prove this relation we assume the equality to be true for $n-1$:

$$(R_1R_2)^{2(n-1)}R_k\omega^{n-1}=\omega^{n-1}R_k. \tag{14}$$

Then we apply ω to both sides of the equation and take into account that ω commutes with R_1R_2:

$$(R_1R_2)^{2(n-1)}\omega R_k\omega^{n-1}=\omega^nR_k \tag{15}$$

and also

$$(R_1R_2)^{2(n-1)}(R_1R_2)^2R_k\omega^n=\omega^nR_k \tag{16}$$

as desired.

In the Penrose pattern case we have that R_k commutes with θ. By induction on n we have $R_k\theta^n=\theta^nR_k$.

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